

RANDOMIZED BLOCKS WITH UNBALANCED DATA

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ABSTRACT

The best linear unbiased estimator (BLUE) of a treatment effect is obtained for randomized block experiments having unequal numbers of observations on the treatments and in which block effects are considered as random. Application to balanced incomplete blocks is also considered.

1. INTRODUCTION

Randomized complete block experiments involve responses from each of a treatments used in each of b blocks, with say, n observations on each treatment in each block. In many experiments $n = 1$. Special cases of block experiments are balanced incomplete blocks in which (with $n = 1$) only $k < a$ treatments are used in each block, such that each treatment is used in r blocks and each treatment pair occurs in λ blocks. In between these extremes of complete blocks and balanced incomplete blocks is the more general case where k_j treatments are used in block j , with $n_{ij} \geq 0$ observations on treatment i in that block and $n_{ij} > 0$ for k_j values of i . John (1971, p.228) considers this general case but only when treating block effects as fixed effects. This paper deals with the case of block effects being treated as random: estimators of treatment effects are derived, along

with their variances, and the results are applied to balanced incomplete blocks to yield an expression for the inter- and intra-block estimator of a treatment effect that is simpler than that given by Kempthorne (1952), Federer (1955), Cochran and Cox (1957), Scheffé (1959) and John (1971).

We use a linear model that has equation

$$y_{ijp} = \tau_i + \beta_j + e_{ijp} \quad (1)$$

where y_{ijp} is the p 'th observation on the i 'th treatment in the j 'th block, for $i = 1, \dots, a$, $j = 1, \dots, b$ and $p = 1, \dots, n_{ij}$ with $n_{ij} \geq 0$, there being at least one $n_{ij} > 0$ for each value of i and for each value of j . In (1), τ_i is the effect due to treatment i , β_j is the effect due to block j and the e_{ijp} s are random errors having mean zero and variance σ_e^2 , and all such terms are uncorrelated. The τ_i s are considered as fixed effects and the point of interest is to estimate contrasts (linear combinations of differences $\tau_i - \tau_h$) among the τ_i s, and to derive sampling variances of those estimators. Two interpretations for the β_j s are available. One is that they are fixed effects, and contrasts among them can be estimated, just as with the treatment effects; and the other is that the β_j s are random effects, which must be taken into account in estimating treatment contrasts.

Randomized complete blocks have the same number of observations on every treatment in every block, i.e., all $n_{ij} = n$. The best linear unbiased estimator (BLUE) of $\tau_i - \tau_h$ is then

$$\text{BLUE}(\tau_i - \tau_h) = \bar{y}_{i..} - \bar{y}_{h..}$$

whether the block effects are treated as fixed or random. This is well known. But with unbalanced data, i.e., not necessarily equal numbers of observations on the treatments either within a block or from block to

block, the estimators of treatment differences are not the simple differences between observed treatment means of balanced data. Furthermore, with unbalanced data, the estimators of treatment differences when block effects are treated as fixed are not the same as when block effects are treated as random. It is these two sets of estimators which we specify in this paper.

2. BLOCKS TREATED AS FIXED EFFECTS

When the β_j s, as well as the τ_i s, are taken as fixed effects, the analysis of (1) for unbalanced data is simply the standard analysis of a 2-way crossed classification fixed effects model without interaction as detailed, for example, in Searle (1971, Section 7.2). Similar, but not such detailed, results are in John (1971, Section 11.5).

Using customary dot and bar notation for totals and means, e.g., $y_{i..} = \sum_{j=1}^b \sum_{p=1}^{n_{ij}} y_{ijp}$ and $\bar{y}_{i..} = y_{i..}/n_{i.}$, the essence of the estimation procedure, taken from Searle [1971, Section 7.2d(iii)] is as follows. First, calculate

$$\tilde{T} = \{t_{ii'}\} \text{ for } i, i' = 1, 2, \dots, a-1 \quad (2)$$

and

$$\tilde{u} = \{u_i\} \text{ for } i = 1, 2, \dots, a-1 \quad (3)$$

with

$$t_{ii} = n_{i.} - \sum_{j=1}^b n_{ij}^2 / n_{.j} \quad (4)$$

$$t_{ii'} = - \sum_{j=1}^b n_{ij} n_{i'j} / n_{.j} \text{ for } i \neq i' \quad (5)$$

and

$$u_i = y_{i..} - \sum_{j=1}^b n_{ij} \bar{y}_{.j}. \quad (6)$$

Define $\tau_a^o = 0$ and $\tau_{\star}^o = \{\tau_i^o\}$ for $i=1, \dots, a-1$, and calculate

$$\tau^o = \begin{bmatrix} \tau_{\star}^o \\ \tau_a^o \end{bmatrix} = \begin{bmatrix} T^{-1}u \\ 0 \end{bmatrix} \quad (7)$$

Then the BLUE of the treatment difference $\tau_i - \tau_h$ for $i \neq h$ is

$$\text{BLUE}(\tau_i - \tau_h) = \tau_i^o - \tau_h^o \quad (8)$$

for τ_i^o and τ_h^o as elements of (7).

Any contrast among the τ_i s is

$$\lambda' \tau = \sum \lambda_i \tau_i \text{ with } \lambda_i \text{ such that } \lambda' 1 = \sum \lambda_i = 0 \quad (9)$$

Its BLUE is

$$\text{BLUE}(\lambda' \tau) = \lambda' \tau^o \quad (10)$$

with sampling variance

$$v[\text{BLUE}(\lambda' \tau)] = v(\lambda' \tau^o) = \lambda' T^{-1} \lambda \sigma_e^2 \quad (11)$$

[This variance result is adapted directly from Searle (1971), using equation (43) on page 182, and (21) on page 268.] And σ_e^2 can be estimated unbiasedly by

$$\hat{\sigma}_e^2 = \frac{\sum_i \sum_j \sum_p y_{ijp}^2 - \sum_j n_{.j} \bar{y}_{.j}^2 - u' T^{-1} u}{N - a - b + 1} \quad (12)$$

On assuming normality, hypotheses about $\lambda' \tau$ can be tested using (10), (11) and (12) in the usual t-statistic. And any composite hypothesis $H: K' \tau = m$, for K' of full row rank less than a , and with $K' 1 = 0$, can be tested using

$$F = (K' \tau^o - m)' (K' T^{-1} K)^{-1} (K' \tau^o - m) / \hat{\sigma}_e^2 r_K \quad (13)$$

an F-statistic on r_K (rank of K) and $N - a - b + 1$ degrees of freedom.

3. BLOCKS TREATED AS RANDOM EFFECTS

3.1 BLU estimation

Treating blocks as random involves taking the β_j s as random variables, all with zero mean, variance σ_β^2 , and uncorrelated with each other and with the e_{ijp} s. This is the mixed model form of block designs, and gives rise to the BLUE of $\tau_i - \tau_h$ being different from (8).

Define $z_{jip} = y_{ijp}$ and \underline{z} as the vector of elements z_{jip} arrayed in lexicon order by p , within i , within j . (Although arraying the data as a vector \underline{y} of elements y_{ijp} ordered by p within j within i is more customary, conveniences to the ensuing algebra arise from using \underline{z} as defined.) With $\underline{\tau}$ and $\underline{\beta}$ being vectors of the τ_i s and β_j s, write (1) in vector form

$$\underline{z} = \underline{X}\underline{\tau} + \underline{Z}\underline{\beta} + \underline{e} \quad , \quad (14)$$

where \underline{X} and \underline{Z} are the appropriate incidence matrices corresponding to $\underline{\tau}$ and $\underline{\beta}$.

The fixed effects model used in Section 2 has the dispersion matrix of \underline{z} corresponding to (14) being $\text{var}(\underline{z}) = \text{var}(\underline{e}) = \sigma_e^2 \underline{I}$, where \underline{I} is an identity matrix. As a result, the equations that lead to BLUEs of $\alpha_i - \alpha_h$ and $\beta_j - \beta_m$ in that fixed effects model are

$$\begin{bmatrix} \underline{X}'\underline{X} & \underline{X}'\underline{Z} \\ \underline{Z}'\underline{X} & \underline{Z}'\underline{Z} \end{bmatrix} \begin{bmatrix} \underline{\tau}^0 \\ \underline{\beta}^0 \end{bmatrix} = \begin{bmatrix} \underline{X}'\underline{z} \\ \underline{Z}'\underline{z} \end{bmatrix} \quad . \quad (15)$$

These equations are less than full rank, and so have many solutions. One of these solutions (see Searle, 1971, Chapter 7) is $\underline{\tau}^0 = \{\tau_i^0\}$ provided by (7). Thus it is that the BLUE of $\tau_i - \tau_h$ is as given in (8).

The mixed model form of (14), where $\underline{\beta}$ is taken as a vector of random variables with $\text{var}(\underline{\beta}) = \sigma_{\beta}^2 \underline{I}_b$, has

$$\text{var}(\underline{z}) = \sigma_e^2 \underline{I} + \underline{Z} \sigma_{\beta}^2 \underline{I}_b \underline{Z}' = \sigma_e^2 \underline{I} + \sigma_{\beta}^2 \underline{Z} \underline{Z}' \quad (16)$$

Denote this by \underline{V} :

$$\underline{V} = \sigma_e^2 \underline{I} + \sigma_{\beta}^2 \underline{Z} \underline{Z}' \quad (17)$$

Then in the mixed model form of (14) the equations for the BLUE of $\underline{\tau}$ are

$$\underline{X}' \underline{V}^{-1} \underline{X} \underline{\tau} = \underline{X}' \underline{V}^{-1} \underline{z}$$

These equations are of full rank, and so in the mixed model

$$\text{BLUE}(\underline{\tau}) = \underline{\tau}^* = (\underline{X}' \underline{V}^{-1} \underline{X})^{-1} \underline{X}' \underline{V}^{-1} \underline{z} \quad (18)$$

We proceed to find $\underline{\tau}^*$ in a more explicit form.

3.2 Incidence matrices

First, the general forms of \underline{X} and \underline{Z} are specified, beginning with an example.

Example Suppose the numbers of observations on three treatments using four blocks are as follows.

n_{ij}					
Treatment	Block				$n_{i.}$
	$j = 1$	$j = 2$	$j = 3$	$j = 4$	
i					
1	4	2	1	6	13
2	4	-	7	-	11
3	1	-	5	4	10
$n_{.j}$	9	2	13	10	34

then in (14)

$$\tilde{X} = \begin{bmatrix} \tilde{l}_4 & \cdot & \cdot \\ \cdot & \tilde{l}_4 & \cdot \\ \cdot & \cdot & \tilde{l}_1 \\ \hline \tilde{l}_2 & \cdot & \cdot \\ \hline \tilde{l}_1 & \cdot & \cdot \\ \cdot & \tilde{l}_7 & \cdot \\ \cdot & \cdot & \tilde{l}_5 \\ \hline \tilde{l}_6 & \cdot & \cdot \\ \cdot & \cdot & \tilde{l}_4 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} \quad \text{and} \quad \tilde{Z} = \begin{bmatrix} \tilde{l}_9 & \cdot & \cdot & \cdot \\ \cdot & \tilde{l}_2 & \cdot & \cdot \\ \cdot & \cdot & \tilde{l}_{13} & \cdot \\ \cdot & \cdot & \cdot & \tilde{l}_{10} \end{bmatrix} \quad (19)$$

where dots in a matrix represent null submatrices of appropriate orders, \tilde{l}_m is a vector of order m with all elements unity, and where \tilde{X} has been partitioned into b submatrices X_j , corresponding to the b blocks in the data.

It is easily seen from (19) that \tilde{Z} is

$$\tilde{Z} = \sum_{j=1}^b \tilde{l}_{n_j} \quad (20)$$

the direct sum of \tilde{l} -vectors of order n_j . X_1 in (20) is of the same nature:

$$X_1 = \begin{bmatrix} \tilde{l}_4 & \cdot & \cdot \\ \cdot & \tilde{l}_4 & \cdot \\ \cdot & \cdot & \tilde{l}_1 \end{bmatrix} = \sum_{i=1}^a \tilde{l}_{n_{i1}} \quad (21)$$

But

$$X_2 = \begin{bmatrix} \tilde{l}_2 & \cdot & \cdot \end{bmatrix} \quad \text{and} \quad X_4 = \begin{bmatrix} \tilde{l}_6 & \cdot & \cdot \\ \cdot & \cdot & \tilde{l}_4 \end{bmatrix}$$

are adaptations of the general form

$$\tilde{X}_j = \sum_{i=1}^a (+) \tilde{1}_{n_{ij}} \quad , \quad (22)$$

derived from adopting the following convention: for $n_{ij} = 0$ define $\tilde{1}_{n_{ij}} = \tilde{1}_0$ as a vector having no rows. This gives, for example,

$$\tilde{X}_4 = \sum_{i=1}^3 (+) \tilde{1}_{n_{i4}} = \begin{bmatrix} \tilde{1}_6 & \cdot & \cdot \\ \cdot & \tilde{1}_0 & \cdot \\ \cdot & \cdot & \tilde{1}_4 \end{bmatrix} = \begin{bmatrix} \tilde{1}_6 & \cdot & \cdot \\ \cdot & \cdot & \tilde{1}_4 \end{bmatrix} \quad . \quad (23)$$

With this convention, \tilde{X}_j is given by (22).

Useful properties of \tilde{X} and \tilde{Z} are as follows: from (19)

$$\tilde{Z}\tilde{Z}' = \sum_{j=1}^b (+) J_{\tilde{n} \cdot j} \quad \text{and} \quad \tilde{Z}'\tilde{Z} = \sum_{j=1}^b (+) n_{\cdot j} \quad , \quad (24)$$

where $J_{\tilde{m}}$ is a square matrix, of order m , with every element unity, and

$\sum_{i=1}^b (+) n_{\cdot j}$ is the diagonal matrix of the numbers of observations in the blocks.

Also

$$\tilde{1}'_{n \cdot j} \tilde{X}_j = [n_{1j} \ n_{2j} \ \cdots \ n_{aj}] = \tilde{c}'_j \quad , \quad \text{say,} \quad (25)$$

where \tilde{c}_j is the $a \times 1$ column vector of the number of observations (including any zeros) in the j 'th block of the data; and

$$\tilde{X}'_j \tilde{X}_j = \sum_{i=1}^a (+) n_{ij} \quad , \quad \text{giving} \quad \tilde{X}' \tilde{X} = \sum_{i=1}^a (+) n_{i \cdot} \quad . \quad (26)$$

3.3 Treatment estimators

$\text{BLUE}(\tau) = \tau^*$ of (18) can now be derived. From (17) and (20)

$$\tilde{Y} = \sigma^2_e \tilde{I}_{n \cdot \cdot} + \sum_{j=1}^b \sigma^2_{\beta} (+) J_{\tilde{n} \cdot j} = \sum_{j=1}^b \left(\sigma^2_e \tilde{I}_{n \cdot j} + \sigma^2_{\beta} J_{\tilde{n} \cdot j} \right)$$

so that

$$\tilde{V}^{-1} \sigma_e^2 = \sum_{j=1}^b (+) \left(\tilde{I}_{n \cdot j} - \frac{\sigma_\beta^2}{\sigma_e^2 + n \cdot j \sigma_\beta^2} J_{n \cdot j} \right) = \sum_{j=1}^b (+) \left(\tilde{I}_{n \cdot j} - \frac{\rho}{1 + n \cdot j \rho} J_{n \cdot j} \right) \quad (27)$$

for

$$\rho = \sigma_\beta^2 / \sigma_e^2 \quad . \quad (28)$$

Hence for (18)

$$\begin{aligned} \tilde{X}' \tilde{V}^{-1} \tilde{X} \sigma_e^2 &= \tilde{X}' \tilde{X} - \tilde{X}' \left[\sum_{j=1}^b (+) \frac{\rho}{1 + n \cdot j \rho} J_{n \cdot j} \right] \tilde{X} \\ &= \sum_{i=1}^a (+) n_{i \cdot} - \sum_{j=1}^b \frac{\rho}{1 + n \cdot j \rho} \tilde{X}' J_{n \cdot j} \tilde{X}_j \end{aligned}$$

and since $\tilde{X}' J_{n \cdot j} \tilde{X}_j = (\tilde{l}'_{n \cdot j} \tilde{X}_j)' \tilde{l}'_{n \cdot j} \tilde{X}_j = \tilde{c}_j \tilde{c}'_j$ from (25) ,

$$\tilde{X}' \tilde{V}^{-1} \tilde{X} \sigma_e^2 = \sum_{i=1}^a (+) n_{i \cdot} - \sum_{j=1}^b \frac{\rho}{1 + n \cdot j \rho} \tilde{c}_j \tilde{c}'_j \quad . \quad (29)$$

Similarly,

$$\begin{aligned} \tilde{X}' \tilde{V}^{-1} \tilde{z} \sigma_e^2 &= \tilde{X}' \tilde{z} - \tilde{X}' \left[\sum_{j=1}^b (+) \frac{\rho}{1 + n \cdot j \rho} J_{n \cdot j} \right] \tilde{z} \\ &= \left\{ z_{\cdot i \cdot} \right\}_{i=1}^{i=a} - \left[\frac{\rho}{1 + n \cdot 1 \rho} \tilde{X}'_1 J_{n \cdot 1} \cdots \frac{\rho}{1 + n \cdot b \rho} \tilde{X}'_b J_{n \cdot b} \right] \tilde{z} \end{aligned}$$

where $\{z_{\cdot i \cdot}\}_{i=1}^{i=a}$ represents an $a \times 1$ vector of elements $z_{\cdot i \cdot}$. Using (25),

$$\tilde{X}'_j J_{n \cdot j} = \tilde{X}'_j \tilde{l}'_{n \cdot j} \tilde{l}'_{n \cdot j} = \tilde{c}_j \tilde{l}'_{n \cdot j}$$

and so, on recalling that elements z_{jip} of \tilde{z} are ordered by p within i within j , we have

$$\tilde{X}' \tilde{V}^{-1} \tilde{z} \sigma_e^2 = \left\{ y_{i \cdot \cdot} \right\}_{i=1}^{i=a} - \sum_{j=1}^b \frac{\rho}{1 + n \cdot j \rho} \tilde{c}_j y_{\cdot j \cdot} \quad . \quad (30)$$

Substituting (29) and (30) into (18) gives

$$\begin{aligned} \tau^* &= \left[\begin{matrix} a \\ (+) \end{matrix} n_{i.} - \sum_{j=1}^b \frac{\rho}{1+n_{.j}\rho} c_j c_j' \right]^{-1} \left[\{y_{i..}\} - \sum_{j=1}^b \frac{\rho y_{.j.}}{1+n_{.j}\rho} c_j \right] \\ &= \left\{ \begin{bmatrix} n_{1.} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & n_{a.} \end{bmatrix} - \sum_{j=1}^b \frac{\rho}{1+n_{.j}\rho} \begin{bmatrix} n_{1j}^2 & n_{1j}n_{2j} & & n_{1j}n_{aj} \\ n_{2j}n_{1j} & n_{2j}^2 & \cdots & n_{2j}n_{aj} \\ \vdots & \vdots & & \vdots \\ n_{aj}n_{1j} & n_{aj}n_{2j} & \cdots & n_{aj}^2 \end{bmatrix} \right\}^{-1} \\ &\quad \times \left\{ \begin{bmatrix} y_{1..} \\ y_{2..} \\ \vdots \\ y_{a..} \end{bmatrix} - \sum_{j=1}^b \frac{\rho y_{.j.}}{1+n_{.j}\rho} \begin{bmatrix} n_{1j} \\ n_{2j} \\ \vdots \\ n_{aj} \end{bmatrix} \right\}. \end{aligned} \quad (31)$$

Since from (16), (17) and (18) the dispersion matrix of τ^* is $\text{var}(\tau^*) = (\tilde{X}'\tilde{V}^{-1}\tilde{X})^{-1}$, it is from (31)

$$D_{\tau^*} = \text{var}(\tau^*) = \left[\begin{matrix} a \\ (+) \end{matrix} n_{i.} - \sum_{j=1}^b \frac{\rho}{1+n_{.j}\rho} c_j c_j' \right]^{-1} \sigma_e^2. \quad (32)$$

3.4 Generality

The generality of results (31) and (32) merits emphasis. Those results apply for any block designs with model equation (1) and the β s and e s random. No matter what the pattern of treatments used in the blocks is, nor the pattern of numbers of observations taken on those treatments, (31) and (32) apply. In all cases n_{ij} is the number of observations on treatment i in

block j , with $n_{ij} = 0$ whenever treatment i is not used in block j . Thus (31) and (32) can be used whether there is a pattern to the n_{ij} s, such as with balanced incomplete blocks, or whether there is no pattern.

3.5 Special cases

It is instructive to see that (31) reduces to the well known results of certain special cases.

(i) $\sigma_\beta^2 = 0$. This means the model is $y_{ijp} = \tau_i + e_{ijp}$. Then $\rho = 0$ and (31) gives $\tau_i^* = \bar{y}_{i..}$, as is to be expected.

(ii) *Balanced data*, all $n_{ij} = n$. The inverse matrix (31) simplifies to

$$\left(bn\tilde{I}_a - \frac{b\rho n^2}{1+\rho n} \tilde{J}_a \right)^{-1} = \frac{1}{bn} \left(\tilde{I}_a + \rho n \tilde{J}_a \right). \quad (33)$$

Thus (31) yields

$$\tau_i^* = \frac{1}{bn} \left[y_{i..} + \rho n y_{...} - \frac{\rho n}{1+\rho n} y_{...} (1 + \rho n) \right] = \bar{y}_{i..}$$

with variance, from (32) and (33) being

$$v(\tau_i^*) = \frac{1}{bn} (1 + \rho n) \sigma_e^2 = (\sigma_e^2 + n\sigma_\beta^2)/bn = v(\bar{y}_{i..}),$$

in the mixed model.

(iii) *The fixed effects model*, when $\sigma_\beta^2 \rightarrow \infty$. Then $\rho \rightarrow \infty$, and in (31) the terms $\rho/(1+n_{.j}\rho) \rightarrow 1/n_{.j}$. This has the effect of making $\tilde{\tau}^*$ of (31) be $\tilde{\tau}^* = \tilde{T}^{*-1} \tilde{u}^*$, where \tilde{T}^* is \tilde{T} of (2) augmented by an a 'th row and column of elements t_{aa} and t_{ia} as given by (4) and (5); and \tilde{u}^* is $\tilde{u}^* = [\tilde{u}' \quad u_a']'$ for u_a given by (3). But then one finds that $\tilde{T}^* \tilde{1} = \tilde{0}$, so \tilde{T}^{*-1} does not exist. Hence

$$\tilde{\tau}^* = \tilde{T}^{*-1} \tilde{u}^* = \begin{bmatrix} \tilde{T}^{-1} & \tilde{0} \\ \tilde{0} & 0 \end{bmatrix} \begin{bmatrix} \tilde{u} \\ u_a \end{bmatrix} = \begin{bmatrix} \tilde{T}^{-1} \tilde{u} \\ 0 \end{bmatrix} = \tilde{\tau}^o$$

of (7). This, too, is as one would expect.

(iv) *Balanced incomplete blocks.* These are a special case of block designs that impose particular patterns of values on the n_{ij} values. One broad class of such patterns is considered in Section 4.

3.6 Estimating variance components

With the fixed effects model, estimation of treatment contrasts, as in (2)-(8), does not involve σ_β^2 . In contrast, the estimator (31) with the mixed model demands a value for $\rho = \sigma_\beta^2/\sigma_e^2$. Unless some a priori value of ρ is acceptable, ρ must be estimated, and this introduces the need for estimating variance components from unbalanced data, with all the attendant difficulties, especially those relating to which method of estimation should be used. Available contenders include maximum likelihood and restricted maximum likelihood (both of which are available from the statistical computing package BMDP4V, see, for example, Searle and Grimes, 1980) and Henderson's Methods II and III. The maximum likelihood methods require iterative solution of non-linear equations but in this simple case of only one variance component other than σ_e^2 estimators obtained by Henderson's method III are available explicitly from Searle [1971, p. 466, equations (117) and (118)]. Thus

$$\hat{\sigma}_e^2 = \frac{\sum_i \sum_j \sum_p y_{ijp}^2 - \sum_j n_{.j} \bar{y}_{.j}^2 - \tilde{u}' \tilde{T}^{-1} \tilde{u}}{n_{..} - a - b + 1}, \quad (34)$$

as in (12); and

$$\hat{\sigma}_\beta^2 = \frac{\tilde{u}' \tilde{T}^{-1} \tilde{u} - \hat{\sigma}_e^2 (a-1)}{n_{..} - \sum_i (\sum_j n_{ij}^2 / n_{i.})}. \quad (35)$$

The denominator of (35) is derived as

$$\begin{aligned}
 \text{tr}[\underset{\sim}{Z}'\underset{\sim}{Z} - \underset{\sim}{Z}'\underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X}) - \underset{\sim}{X}'\underset{\sim}{Z}] &= \text{tr} \left(\underset{\sim}{\underset{\sim}{j=1}}^b \underset{\sim}{\underset{\sim}{(+)}} \underset{\sim}{n_{\cdot j}} \right) - \text{tr} \left[\underset{\sim}{Z}'\underset{\sim}{X} \left(\underset{\sim}{\underset{\sim}{i=1}}^a \underset{\sim}{\underset{\sim}{(+)}} \frac{1}{\underset{\sim}{n_{i \cdot}}} \right) \underset{\sim}{X}'\underset{\sim}{Z} \right] \\
 &= n_{\cdot \cdot} - \text{tr}[\underset{\sim}{Z}'\underset{\sim}{X}D(\underset{\sim}{Z}'\underset{\sim}{X}D)'] \text{ for } D = \underset{\sim}{\underset{\sim}{i=1}}^a \underset{\sim}{\underset{\sim}{(+)}} 1/\sqrt{\underset{\sim}{n_{i \cdot}}} \\
 &= n_{\cdot \cdot} - (\text{sum of squares of elements of } \underset{\sim}{Z}'\underset{\sim}{X}D) \\
 &= n_{\cdot \cdot} - \sum_j (\text{sum of squares of elements of } \underset{\sim}{c_j}'D) \\
 &= n_{\cdot \cdot} - \sum_i \sum_j \left(\underset{\sim}{n_{ij}} / \sqrt{\underset{\sim}{n_{i \cdot}}} \right)^2 \\
 &= n_{\cdot \cdot} - \sum_i \left(\sum_j \underset{\sim}{n_{ij}}^2 \right) / \underset{\sim}{n_{i \cdot}}.
 \end{aligned}$$

The estimators (34) and (35) yield $\hat{\rho} = \hat{\sigma}_\beta^2 / \hat{\sigma}_e^2$. Denote $\underset{\sim}{\tau}^*$ of (32) by $\underset{\sim}{\tau}^*(\rho)$, and replace ρ therein by $\hat{\rho}$ to yield $\underset{\sim}{\tau}^*(\hat{\rho})$. Then the Kackar and Harville (1981) results indicate that $\underset{\sim}{\tau}^*(\hat{\rho})$ is an unbiased estimator of $\underset{\sim}{\tau}$. Similarly, denote (32) by $\underset{\sim}{D}_{\underset{\sim}{\tau}^*}(\rho, \sigma_e^2)$; then replacing ρ and σ_e^2 therein by $\hat{\rho}$ and $\hat{\sigma}_e^2$ yields $\underset{\sim}{D}_{\underset{\sim}{\tau}^*}(\hat{\rho}, \hat{\sigma}_e^2)$ which can be used as an estimate of the dispersion matrix of $\underset{\sim}{\tau}^*(\hat{\rho})$. Kackar and Harville (1984) indicate how this estimate can be improved.

4. BALANCED INCOMPLETE BLOCKS

4.1 Specification

Balanced incomplete blocks are special cases of block designs with unbalanced data that are of particular interest. Although in their most general form balanced incomplete block designs do not have to be confined to those wherein there is either only one or no observation on each treatment in each block, this is certainly the most common form of the design appearing in the literature and we too confine attention to that form. But

it is to be emphasized that (31) and (32), through their generality vis-à-vis unbalanced data, do provide opportunity for considering balanced (and partially balanced) incomplete blocks of far broader form than considered here.

The usual characteristics of a balanced incomplete block design involving $a = t$ treatments in b blocks are that only $k < t$ treatments are used in each block, that each treatment is used in only $r < b$ blocks and that every pair of treatments occurs in precisely λ different blocks. Thus

$$tr = bk \quad \text{and} \quad \lambda(t-1) = r(k-1) . \quad (36)$$

Confining attention to the $n_{ij} = 0$ or 1 case, thus gives

$$n_{i.} = r \quad \text{and} \quad n_{.j} = k$$

and also permits dropping the third subscript from y_{ijp} to denote the response from treatment i in block j as y_{ij} . Then with

$$\bar{y}_{i.} = \sum_{j=1}^b y_{ij}/r \quad \text{and} \quad \bar{y}_{.j} = \sum_{i=1}^a y_{ij}/k$$

and denoting by \bar{y}_i^* the mean of the block means for the blocks containing treatment i , we have

$$\bar{y}_i^* = \frac{1}{r} \sum_{j=1}^b n_{ij} \bar{y}_{.j} = \frac{1}{rk} \sum_{j=1}^b n_{ij} y_{.j} = \frac{1}{rk} y_i^* . \quad (37)$$

4.2 Estimated treatment effects

The preceding notation immediately simplifies (31) to be

$$\hat{\tau}^* = \left(rI_a - \frac{\rho}{1+k\rho} \sum_{j=1}^b \underline{c}_j \underline{c}_j' \right)^{-1} \left\{ y_{i.} - \frac{\rho}{1+k\rho} \sum_{j=1}^b n_{ij} y_{.j} \right\} .$$

With \underline{c}_j being, from (25), the vector of numbers of observations (0s and 1s) on the treatments in block j , $\underline{c}_j \underline{c}_j'$ has elements that are 0s and 1s, with 1s in the diagonal and as element i, i' ($i \neq i'$) for those treatments i and i' that both occur in block j . Thus, $\sum_{j=1}^b \underline{c}_j \underline{c}_j'$ has diagonal elements r and

off-diagonal elements λ . Therefore

$$\tau_i^* = \left[r\tilde{I}_t - \frac{\rho}{1+k\rho} \left(\{r-\lambda\} \tilde{I}_t + \lambda \tilde{J}_t \right) \right]^{-1} \left\{ r\bar{y}_{i.} - \frac{\rho}{1+k\rho} rky_i^* \right\}_{i=1}^{i=t} \quad (38)$$

and after some straightforward simplification this reduces, for $i = 1, \dots, t$ to

$$\tau_i^* = \frac{1}{r+\lambda t\rho} \left[r(1+k\rho)\bar{y}_{i.} - rk\rho\bar{y}_i^* + \lambda t\rho\bar{y}_{..} \right] \quad (39)$$

And, from (32 and (38),

$$\text{var}(\tau_i^*) = \left[r\tilde{I}_t - \frac{\rho}{1+k\rho} \left(\{r-\lambda\} \tilde{I}_t + \lambda \tilde{J}_t \right) \right]^{-1} \sigma_e^2 = \frac{1+k\rho}{r+\lambda t\rho} (\tilde{I}_t + \rho\lambda\tilde{J}_t/r) \sigma_e^2 \quad .$$

This gives the variance of τ_i^* and the covariance of τ_i^* and τ_h^* for $i \neq h$ as

$$v(\tau_i^*) = \frac{(1+k\rho)}{r+\lambda t\rho} (1 + \rho\lambda/r) \sigma_e^2 = \frac{r+\lambda\rho}{r(r+\lambda t\rho)} (\sigma_e^2 + k\sigma_\beta^2) \quad (40)$$

and

$$\text{cov}(\tau_i^*, \tau_h^*) = \frac{\lambda\rho}{r(r+\lambda t\rho)} (\sigma_e^2 + k\sigma_\beta^2) \quad (41)$$

Also, a treatment difference is estimated by

$$\tau_i^* - \tau_h^* = \frac{1}{r+\lambda t\rho} [r(1+k\rho)(\bar{y}_{i.} - \bar{y}_{h.}) - rk\rho(\bar{y}_i^* - \bar{y}_h^*)] \quad (42)$$

with variance

$$v(\tau_i^* - \tau_h^*) = \frac{2}{r+\lambda t\rho} (\sigma_e^2 + k\sigma_\beta^2) \quad (43)$$

4.3 Reconciliation with inter-intra block estimators

The treatment and treatment-difference estimators in (39) and (42) are precisely the same as those given in at least five well-known texts. Nevertheless, since text notations are not all the same, and some are far less succinct than that of (39) and (42), we indicate how those text results can be reconciled with (39) and (42).

(i) **Kemphorne (1952, Section 26.4).** The V_j and T_j he uses are, respectively, our $y_{i.}$ and y_i^* ; and with $W = 1/\sigma_e^2$ and $W' = 1/(\sigma_e^2 + k\sigma_\beta^2)$, his v is

$$v = \frac{W - W'}{Wt(k-1) + W'(t-k)} = \frac{r\rho}{(t-1)(r+\lambda t\rho)}$$

Then the y_j of his (11) is

$$\frac{V_i}{r} + \frac{v}{r} [(t-k)V_j - (t-1)T_j] = \tau_i^* - \frac{\lambda t\rho}{r+\lambda t\rho} \bar{y}..$$

so that the difference between two of Kempthorne's y_j 's is the difference between two τ_i^* s.

(ii) Federer (1955, page 418) Equation (XIII-14) shows an adjusted treatment mean as $\bar{x}'_j = \bar{x}_j + \mu W_j/r$ for \bar{x}_j being $\bar{y}_{i.}$, μ being Kempthorne's v and, from (XIII-7),

$$W_j = (t-k)y_{i.} - (t-1)y_i^* + (k-1)y_{..}$$

μ comes from the first part of (XIII-18) as

$$\mu = \frac{w - w'}{wt(k-1) + w'(t-k)}$$

with $w = 1/E_e$ from (XIII-16) and $E_e = \sigma_e^2$ from the table at the bottom of page 416. But w' is given as

$$w' = \frac{t(r-1)}{k(b-1)E_b - (t-k)E_e}$$

in (XIII-17). It is only when

$$E_b = \sigma_e^2 + \frac{kb-t}{b-1} \sigma_\beta^2$$

is used, also from the table at the bottom of page 416, that w' reduces to $1/(\sigma_e^2 + k\sigma_\beta^2)$, μ reduces to Kempthorne's v and then \bar{x}'_j becomes τ_i^* .

(iii) Cochran and Cox (1957, p.444). Using $E_b = \sigma_e^2 + [k(r-1)/r]\sigma_\beta^2$ and $E_e = \sigma_e^2$ in their μ in paragraph 4 on page 445 is

$$\mu = \frac{r(E_b - E_e)}{rt(k-1)E_b + k(b-r-t+1)E_e} = \frac{r\rho}{(t-1)(r+\lambda t\rho)}$$

Then, with $T = y_{i.}$, $B_t = y_i^*$ and $G = y_{..}$, their W on page 444 is

$$W = (t-k)y_{i.} + (t-1)y_i^* + (k-1)y_{..}$$

and their adjusted treatment mean from page 445 is

$$(T + \mu W) = \tau_i^* .$$

(iv) **Sheffé (1959, pages 165-178).** His notation is so different from that of most other writers that it is necessary to display equivalent notations.

	<u>Scheffé</u>	<u>Here</u>
p.161	I, J, r, k	t, b, r, k , respectively
p.162	k_{ij}	n_{ij}
p.164	g_i, h_j, G_i	$y_{i.}, y_{.j}, y_{i.} - r\bar{y}_i^*$, respectively
(5.2.10)	T_i	$y_i^* = kry_i^*$
(5.2.17)	$\delta = \frac{rk-r+\lambda}{rk} = \frac{(k-1)I}{k(I-1)}$	$\frac{\lambda t}{rk} = \frac{(k-1)t}{k(t-1)}$
(5.2.18)	$\hat{\alpha}_i = G_i/r\delta$	$rk(\bar{y}_{i.} - \bar{y}_i^*)/\lambda t$
(5.2.33)	$\alpha'_i = \frac{T_i - rJ^{-1}\sum_j h_j}{r - \lambda}$	$\frac{rk(\bar{y}_i^* - \bar{y}_{..})}{r - \lambda}$
(5.2.32b)	$\sigma_f^2 = k^2\sigma_B^2 + k\sigma_e^2$	$k(\sigma_e^2 + k\sigma_\beta^2)$
p.172	$\psi = \sum c_i \alpha_i$ and $\psi' = \sum c_i \alpha'_i$, $\sum c_i = 0$.	
p.174	$w = r\delta/\sigma_e^2$	$\lambda t/k\sigma_e^2$
	$w' = (r-\lambda)/\sigma_f^2$	$(r-\lambda)/[k(\sigma_e^2 + k\sigma_\beta^2)]$
p.175	$\psi^* = \frac{w\psi + w'\psi'}{w + w'}$	

ψ^* is described by Scheffé as being unbiased and having minimum variance. It therefore corresponds to a τ_i^* . Since ψ is a contrast of α_i s it is also a contrast of $(\mu + \alpha_i)$ -terms of Scheffé's model. To make ψ^* consistent with τ_i^* we therefore consider

$$\psi_i^* = \frac{w(\hat{\mu} + \hat{\alpha}_i) + w'(\hat{\mu}' + \hat{\alpha}_i')}{w + w'}$$

Scheffé gives $\hat{\alpha}_i$ on page 165, but nowhere shows a corresponding $\hat{\mu}$, the only intimation being the last line of page 164 in "correction term for the grand mean." From this we infer $\hat{\mu} = \bar{y}_{..}$; and from (5.2.33), $\hat{\mu}' = \bar{y}_{..}$. Then ψ_i^* simplifies to τ_i^* .

(v) John (1971, pages 224-234). This author defines

$$\psi^* = \frac{w_1 \hat{\psi} + w_2 \tilde{\psi}}{w_1 + w_2}$$

on page 236, with, from the first line of page 237,

$$w_1 = \frac{\lambda t}{k\sigma^2}, \quad w_2 = \frac{r-\lambda}{\sigma_f^2}, \quad \text{where } \sigma_f^2 = k(\sigma_e^2 + k\sigma_\beta^2)$$

On page 234 are

$$T_i = y_{i.}, \quad B_j = y_{.j} \quad \text{and} \quad T'_i = y_i^*$$

Then from page 224

$$\hat{\psi} = (k/\lambda t) \sum c_i Q_i \quad \text{for } \sum c_i = 0 \quad \text{and} \quad Q_i = kT_i - T'_i$$

and from page 236

$$\tilde{\psi} = \sum c_i T'_i / (r-\lambda)$$

With these substitutions

$$\psi^* = \frac{(1+k\rho)r\sum c_i(\bar{y}_{i.} - \bar{y}_i^*) + r\sum c_i \bar{y}_i^*}{r + \lambda t\rho} = \frac{(1+k\rho)r\sum c_i \bar{y}_{i.} - r k\rho \sum c_i \bar{y}_i^*}{r + \lambda t\rho}$$

Thus if for some particular i and $h \neq i$ we take $c_i = 1$ and $c_h = -1$ and all other c s zero, then

$$\psi^* = \frac{1}{r+\lambda t \rho} \left[r(1+k\rho)(\bar{y}_{i.} - \bar{y}_{h.}) - rk\rho(\bar{y}_i^* - \bar{y}_n^*) \right]$$

which is $\tau_i^* - \tau_h^*$ of (42).

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